Almost-Periodic Solutions of Navier-Stokes Equations and Inequalities

By GIOVANNI PROUSE

Introduction

In this talk I would like to present some results, old and new, concerning almost-periodic solutions of Navier-Stokes equations and inequalities, which govern the motion of viscous compressible or incompressible fluids (respectively gases or liquids).

Of the various problems which can be associated with this motion I shall, in what follows, for the sake of simplicity, consider only the one corresponding to a fluid in a bounded 2- or 3-dimensional domain Ω, which boundary Γ constituted by a material surface. Denoting by $\vec{u}\,(x,t)$ $(x=\{x_1, x_2, x_3\} \in \overline{\Omega})$ the velocity of the fluid, the problem indicated above corresponds, by the limit layer theory, to the homogeneous Dirichlet boundary condition

$$\vec{u}\,(x,t) = 0 \qquad (x \in \Gamma). \tag{1.1}$$

The following notations will be used in the sequel.

$\vec{f}(x,t)$ external force acting on the fluid;

$p(x,t)$ pressure;

$\varrho(x,t)$ density; in the incompressible case (ϱ = const) I shall assume, for simplicity, $\varrho = 1$;

μ, ζ viscosity coefficients (resp. shear and bulk viscosity);

$\mathscr{D}$ space of functions (or vectors) $\in C^\infty(\overline{\Omega})$ and with compact support in Ω;

$\mathscr{N}$ space of vectors $\vec{v} \in \mathscr{D}$ and such that div $\vec{v} = 0$;

H^s (s integer $\geqslant 0$) space of functions (or vectors) square summable in Ω, together with their derivatives (in the sense of distributions) of order $\leqslant s$;

N^s closure of $\mathscr{N}$ in H^s.

The most common mathematical model associated to the motion of a viscous fluid is constituted by the *Navier-Stokes equations* which, in the case of incompressible fluids, take the form

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \mu \Delta \vec{u} + (\vec{u}.\mathrm{grad})\,\vec{u} + \mathrm{grad}\, p = \vec{f} \\ \\ \mathrm{div}\, \vec{u} = 0. \end{cases} \tag{1.2}$$

while, if the fluid is compressible, are expressed by

$$\begin{cases} \varrho \dfrac{\partial \vec{u}}{\partial t} + (\zeta + \dfrac{1}{3}\mu) \text{ grad div } \vec{u} - \mu \Delta \vec{u} + \varrho\, (\vec{u}.\text{grad})\, \vec{u} + \text{grad } p = \varrho \vec{f} \\ \dfrac{\partial \varrho}{\partial t} + \text{div } (\varrho \vec{u}) = 0 \\ p = p(\varrho). \end{cases} \tag{1.3}$$

The third equation of (1.3) is an *equation of state* which, in most practical cases, is given by $p = k\varrho^{\gamma}$ $(k, \gamma > 0)$.

It should be noted that (1.2) cannot be considered as a special case of (1.3), since the two systems are essentially different.

Another model associated to viscous incompressible flow corresponds to the *Navier-Stokes inequalities* which are introduced as follows. Observing that the Navier-Stokes equations are non-relativistic and, consequently, do not have any physical meaning when $|\vec{u}|$ approaches the velocity of light, the model (1.2) is equivalent, from a physical point of view, to the one corresponding to the relationships

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \mu \Delta \vec{u} + (\vec{u}.\text{grad})\, \vec{u} + \text{grad } p = \vec{f} \quad \text{where } |\vec{u}| < c \\ \text{div } \vec{u} = 0 \quad , \quad |\vec{u}| \leqslant c \\ \vec{u} \text{ continuous at the “interfaces” of the two sets in which resp. } |\vec{u}| < c \text{ and } |\vec{u}| = c. \end{cases} \tag{1.4}$$

It is well known, on the other hand, from the theory of differential inequalities (see, for instance, Lions [1]) that (1.4) is equivalent to the system

$$\begin{cases} \displaystyle\int_{t_1}^{t_2} \int_{\Omega} \left(\frac{\partial \vec{u}}{\partial t} - \mu \Delta \vec{u} + (\vec{u}.\text{grad})\, \vec{u} + \text{grad } p - \vec{f} \right) (\vec{u} - \vec{\varphi})\, dt\, d\Omega \leqslant 0. \\ \text{div } \vec{u} = 0 \\ |\vec{u}| \leqslant c \end{cases} \tag{1.5}$$

$\forall\, \vec{\varphi}$ such that $|\vec{\varphi}| \leqslant c$ and $\forall\, t_1, t_2 \in (-\infty, +\infty)$.

System (1.5) therefore constitutes an *inequality model* for the problem considered, in the incompressible case. An analogous model could obviously be given for compressible fluids, but it will not be considered here.

In the next section I shall recall some results concerning the almost-periodic solutions of the three models presented; it is however useful to first briefly summarize the main existence and uniqueness theorems of a solution of (1.2), (1.3), (1.5) satisfying (1.1) and the initial conditions

$$\vec{u}(x, 0) = \vec{u}_0(x) \qquad \text{(incompressible case)}$$
$$\vec{u}(x, 0) = \vec{u}_0(x)\,,\ \varrho(x, 0) = \varrho_0(x) \qquad \text{(compressible case)}$$

These theorems represent, in fact, the first step in the study of the almost-periodic solutions.

The solutions will always be intended in the sense of distributions, while I shall not, for the sake of simplicity, indicated explicitly the functional spaces in which the solutions are found, or the assumptions on the data.

Considering system (1.2), Hopf [2] proved the global (in time) existence of a solution in any space dimension; the uniqueness of such a solution can however be guaranteed only in 2 dimensions (Lions and Prodi [3]). An existence and uniqueness theorem in $\Omega \times (0, T)$, Ω 3-dimensional, holds provided $\vec{f}$ is "sufficiently small" (Kieselev and Ladyzenskaja [4]).

One can, on the other hand, prove a global existence and uniqueness theorem for the solution in $\Omega \times (0, T)$ of (1.5) (Prouse [5]).

In the compressible case, only a local existence and uniqueness theorem holds (Valli [6]); in order to obtain global existence and uniqueness, one must assume that $\vec{f}$ is "sufficiently small" (Marcati and Valli [7]).

Almost-periodicity theorems

The models introduced in the preceding section all correspond to dissipative problems, and the study of their almost-periodic solutions follows therefore from the guidelines given, for ordinary dissipative differential equations, by Favard [8] and Amerio [9] respectively in the linear and non linear case.

In the theory of almost-periodic solutions of partial differential equations, vector valued functions play an essential role, together with the concepts of weakly almost-periodic and S^p-Stepanov almost-periodic functions. For these concepts and for the basic definitions and properties of functions with values in a Banach space, I refer to the note by L. Amerio which appears in the present volume (see also Amerio, Prouse [10]).

While the details of the proofs of the existence and uniqueness of an almost-periodic solution, under the assumption that $\vec{f}(t)$ is almost-periodic, are obviously different for the three models considered, the basic scheme is similar and consists essentially of the following steps:

a) A global existence theorem in $[t_0, +\infty)$;

b) An existence and uniqueness theorem of a solution $\vec{u}(t)$ (or $\{\vec{u}(t), \varrho(t)\}$) bounded on $J = (-\infty, +\infty)$ (assuming $\vec{f}(t)$ bounded on J);

c) The proof that $\vec{u}(t)$ ($\{\vec{u}(t), \varrho(t)\}$) is weakly almost-periodic if $\vec{f}(t)$ is weakly almost-periodic;

d) The proof that the range of $\vec{u}(t)$ ($\{\vec{u}(t), \varrho(t)\}$) is relatively compact if $\vec{f}(t)$ is almost-periodic.

Observe that point a) corresponds essentially to the results recalled in the preceding section, setting $T = +\infty$.

Assuming that $f(t)$ is S^2-Stepanov almost-periodic, the following theorems then hold.

Theorem I (Prouse [11]): *If Ω is 2-dimensional, $\vec{f}(t) \in L^{\infty}(J;L^2)$ and is "sufficiently small",* (1.1), (1.2) *admit a unique solution $\vec{\tilde{u}}(t)$ which is N^0-Bohr and N^1-S^2-Stepanov almost-periodic.*

Theorem II (Foias [12], Heywood [13]): *If Ω is 3-dimensional and of class C^3, $\vec{f}(t) \in L^2_{loc}(J, N^1) \cap H^1_{loc}(J, (N^1)^*)$ and is "sufficiently small", then* (1.1), (1.2) *admit a unique solution $\vec{\tilde{u}}(t)$ which is N^0-Bohr and N^1-S^2-Stepanov almost-periodic.*

Theorem III (Marcati and Valli [7]): *If Ω is 3-dimensional and of class C^4, $p \in C^3$, $p' > 0$, $\vec{f}(t) \in L^2_{loc}(J;H^1) \cap H^1_{loc}(J;H^{-1})$ and is "sufficiently small", then* (1.1), (1.3) *admit a unique solution $\{\vec{\tilde{u}}(t), \tilde{\varrho}(t)\}$ with $\vec{\tilde{u}}(t)$ H^1-Bohr and H^2-S^2-Stepanov almost-periodic, $\tilde{\varrho}(t)$ L^2-Bohr and H^2-S^2-Stepanov almost-periodic.*

Theorem IV (Prouse [14]): *If Ω is 3-dimensional, $\vec{f}(t) \in L^{\infty}(J;L^2)$ and is "sufficiently small", then* (1.1), (1.4) *admit a unique solution $\vec{\tilde{u}}(t)$ which is N^0-Bohr and N^1-S^2-Stepanov almost-periodic.*

References

[1] Lions, J. L.: *Quelques méthodes de resolution des problèmes aux limites non linéaires.* Dunod, 1969.

[2] Hopf, E.: *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen.* Math. Nachr., 4, 1951.

[3] Lions, J. L., Prodi, G.: Un théorème d'existence et d'unicité dans les équations de Navier-Stokes en dimension 2. *C. R. Acad. Sci.*, 248, 1959.

[4] Kieselev, A. A., Ladyzenskaja, O. A.: On the existence and uniqueness of the solution of the non stationary problem for a viscous, incompressible fluid (in Russian). *Isv. Akad. Nauk,* 21, 1957.

[5] Prouse, G.: On an inequality related to the motion of viscous, incompressible fluids, notes I, II. *Rend. Acc. Lincei,* 67, 1979.

[6] Valli, A.: Uniqueness theorems for compressible viscous fluids, especially when the Stokes relation holds. *Boll. Un. Mat. It.*, 18-C, 1981.

[7] Marcati, P., Valli, A.: Almost-periodic solutions to the Navier-Stokes equations for compressible fluids. *Boll. Un. Mat. It.*, 4-D, 1985.

[8] Favard, J.: *Leçons sur les fonctions presque-périodiques.* Gauthier-Villars, 1933.

[9] Amerio, L.: Soluzioni quasi-periodiche o limitate di sistemi differenziali non lineari quasi-periodici o limitati. *Ann. di Mat.*, 39, 1955.

[10] Amerio, L., Prouse, G.: *Almost-periodic functions and functional equations.* Van Nostrand, 1971.

[11] Prouse, G.: Soluzioni quasi-periodiche dell'equazione differenziale di Navier-Stokes in 2 dimensioni. *Rend. Sem. Mat. Padova,* 33, 1963.

[12] Foias, C.: Essais dans l'etude des solutions des équations de Navier-Stokes dans l'espace. L'unicité et la presque-périodicité des solutions petites. *Rend. Sem. Mat. Padova,* 32, 1962.

[13] Heywood, J. G.: The Navier-Stokes equations: on the existence, regularity and decay of solutions. *Indiana Univ. Math. J.*, 29, 1980.

[14] Prouse, G.: Almost-periodic solutions of an inequality related to the motion of viscous, incompressible fluids. Preprint.

Politecnico di Milano
Dipartimento di Matematica
20133 Milano, Piazza Leonardo da Vinci 32
Italy